# 1 强大数定律的证明

这份笔记整理了 Etemadi 在 1981 年给出的强大数定律的证明。Etemadi 版本的大数定律只要求随机变量序列两两独立、同分布、可积，这比 Kolmogorov 大数定律的假设要弱 (相互独立、同分布、可积)。

**Theorem 1.1** (Strong law of large number, Etemadi)**.** *Suppose* $\{X_n : n \geq 1\}$ *is a sequence of pairwise independent identically distributed r.v. with* $\mathbb{E}(|X_i|) < \infty$. *Let* $\mu = \mathbb{E}(X_i)$,

$$S_n = X_1 + X_2 + \cdots + X_n,$$

*then*

$$\frac{S_n}{n} \to \mu \quad a.s. \tag{1}$$

*as* $n \to \infty$.

**Lemma 1.2.** *Suppose* $X$ *is a r.v. with* $X \geq 0$.*, then we have*

1. $$\sum_{n=1}^{\infty} \mathbb{P}(X \geq n) \leq \mathbb{E}(X) \leq 1 + \sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$$

2. *For any* $p > 0$,
$$\mathbb{E}(X^p) = \int_0^{\infty} pt^{p-1}\mathbb{P}(X > t)\,\mathrm{d}t.$$

**Lemma 1.3.** *Let* $Y_k = X_k \mathbb{1}_{\{|X_k| \leq k\}}$ *and* $T_n = Y_1 + Y_2 + \cdots + Y_n$, *then*

$$\frac{T_n}{n} \to \mu \quad a.s.$$

*implies* (1).

*Proof.* 1. By Lemma 1.2,

$$\sum_{k=1}^{\infty} \mathbb{P}(|X_k| > k) = \sum_{n=1}^{\infty} \mathbb{P}(|X_1| > k) \leq \mathbb{E}(|X_1|) < \infty,$$

then by Borel-Cantelli Lemma,

$$\mathbb{P}(X_k \neq Y_k \ i.o.) = \mathbb{P}(|X_k| > k \ i.o.) = 0. \tag{2}$$

2. Let $A = \{X_k \neq Y_k \ i.o.\}$, for any $\omega \in A^c$ (w.p.1.), $X_k \neq Y_k$ only for finitely many $k$, i.e. there exists $N(\omega)$, s.t. $X_k(\omega) = Y_k(\omega)$ for all $k \geq N(\omega)$. Therefore for almost sure $\omega$,

$$\lim_{n\to\infty} \frac{S_n(\omega)}{n} - \lim_{n\to\infty} \frac{T_n(\omega)}{n} = \lim_{n\to\infty} \frac{S_n - T_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{N(\omega)} (X_k(\omega) - Y_k(\omega)) = 0.$$ □

**Lemma 1.4.** *For all $x \geq 0$,*

$$x \sum_{k>x}^{\infty} \frac{1}{k^2} \leq 2.$$

*Proof.* By comparison with the integral[1], we have

$$x \sum_{k>x}^{\infty} \frac{1}{k^2} = x \sum_{k=[x]+1}^{\infty} \frac{1}{k^2} \leq \frac{x}{([x]+1)^2} + x \int_{[x]+1}^{\infty} \frac{1}{t^2} \,\mathrm{d}t = \frac{x}{([x]+1)^2} + \frac{x}{[x]+1} \leq 2.$$

□

**Lemma 1.5.**

$$\sum_{k=1}^{\infty} \frac{\mathbb{E}(Y_k^2)}{k^2} \leq 4\mathbb{E}(|X_1|).$$

*Proof.* By Lemma 1.2,

$$\mathbb{E}(Y_k^2) = \int_0^{\infty} \mathbb{P}(Y_k^2 > t) \,\mathrm{d}t = \int_0^{\infty} 2x\mathbb{P}(|Y_k| > x) \,\mathrm{d}x = \int_0^{k} 2x\mathbb{P}(|Y_k| > x) \,\mathrm{d}x \leq \int_0^{k} 2x\mathbb{P}(|X_1| > x) \,\mathrm{d}x,$$

therefore

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\mathbb{E}(Y_k^2)}{k^2} &\leq \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{\infty} \mathbb{1}_{\{x<k\}} 2x\mathbb{P}(|X_1| > x) \,\mathrm{d}x \\
&= \int_0^{\infty} 2x\mathbb{P}(|X_1| > x) \sum_{k=1}^{\infty} \frac{1}{k^2} \mathbb{1}_{\{x<k\}} \,\mathrm{d}x \\
&= \int_0^{\infty} 2\mathbb{P}(|X_1| > x) \left( x \sum_{k>x}^{\infty} \frac{1}{k^2} \right) \mathrm{d}x \\
&\leq 4 \int_0^{\infty} \mathbb{P}(|X_1| > x) \,\mathrm{d}x \\
&= 4\mathbb{E}(|X_1|).
\end{aligned}$$

□

*Proof of Theorem 1.1.* 1. Since $X_k = X_k^+ - X_k^-$, (1) holds if

$$\frac{\sum_{k=1}^n X_k^+}{n} \to \mathbb{E}(X_1^+) \quad \text{and} \quad \frac{\sum_{k=1}^n X_k^-}{n} \to \mathbb{E}(X_1^-) \quad a.s.$$

hence it is sufficient to prove the case when $X_k \geq 0$.
2. Fix $\alpha > 1$, let $k_n = [\alpha^n]$. Then we will consider the subsequence $\{T_{k_n} : n \geq 1\}$. For any $\varepsilon > 0$, by Chebyshev's inequality, we have

$$\mathbb{P}(|T_{k_n} - \mathbb{E}(T_{k_n})| > \varepsilon k_n) \leq \frac{1}{\varepsilon^2 k_n^2} \mathbb{E}(|T_{k_n} - \mathbb{E}(T_{k_n})|^2) = \frac{\mathrm{Var}(T_{k_n})}{\varepsilon^2 k_n^2},$$

[1] See https://en.wikipedia.org/wiki/Integral_test_for_convergence

thus

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbb{P}(|T_{k_n}-\mathbb{E}(T_{k_n})|>\varepsilon k_n) &\le \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{\mathrm{Var}(T_{k_n})}{k_n^2}\\ &= \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{k_n^2}\sum_{m=1}^{k_n}\mathrm{Var}(Y_m) \quad \text{(by pairwise independence)}\\ &= \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{k_n^2}\mathbb{1}_{\{m\le k_n\}}\mathrm{Var}(Y_m)\\ &= \frac{1}{\varepsilon^2}\sum_{m=1}^{\infty}\mathrm{Var}(Y_m)\sum_{n=1}^{\infty}\frac{1}{k_n^2}\mathbb{1}_{\{m\le k_n\}}\\ &= \frac{1}{\varepsilon^2}\sum_{m=1}^{\infty}\mathrm{Var}(Y_m)\sum_{n:k_n\ge m}^{\infty}\frac{1}{k_n^2}\\ &\le \frac{1}{\varepsilon^2}\sum_{m=1}^{\infty}\mathbb{E}(Y_m^2)\sum_{n:k_n\ge m}^{\infty}\frac{1}{k_n^2}.\end{aligned}$$

Since $k_n=[\alpha^n]\ge\alpha^n/2$, let $n_0=\inf\{n\in\mathbb{Z}_+:[\alpha^n]\ge m\}$, then $\alpha^{n_0}\ge[\alpha^{n_0}]\ge m$, we have

$$\sum_{n:k_n\ge m}^{\infty}\frac{1}{k_n^2}\le\sum_{n=n_0}^{\infty}\frac{4}{\alpha^{2n}}=\frac{4}{\alpha^{2n_0}}\sum_{n=0}^{\infty}\frac{1}{\alpha^{2n}}\le\frac{4}{m^2}\cdot\frac{1}{1-\alpha^{-2}},$$

therefore

$$\sum_{n=1}^{\infty}\mathbb{P}(|T_{k_n}-\mathbb{E}(T_{k_n})|>\varepsilon k_n)\le\frac{4}{\varepsilon^2(1-\alpha^{-2})}\sum_{m=1}^{\infty}\cdot\frac{\mathbb{E}(Y_m^2)}{m^2}\le\frac{16}{\varepsilon^2(1-\alpha^{-2})}\cdot\mathbb{E}(|X_1|)<\infty,$$

then by Borel-Cantelli Lemma,

$$\mathbb{P}\left(\frac{|T_{k_n}-\mathbb{E}(T_{k_n})|}{k_n}>\varepsilon\ i.o.\right)=0,$$

i.e. almost surely, $\dfrac{|T_{k_n}-\mathbb{E}(T_{k_n})|}{k_n}\le\varepsilon$ for all large enough $n$, in other word,

$$\frac{T_{k_n}-\mathbb{E}(T_{k_n})}{k_n}\to 0,\quad a.s.$$

Since $X_1\mathbb{1}_{\{|X_1|\le k\}}\uparrow X_1$, by monotone convergence theorem,

$$\mathbb{E}(Y_k)=\mathbb{E}(X_k\mathbb{1}_{\{|X_k|\le k\}})=\mathbb{E}(X_1\mathbb{1}_{\{|X_1|\le k\}})\to\mathbb{E}(X_1)=\mu,$$

thus by Stolz–Cesàro theorem,

$$\lim_{n\to\infty}\frac{\mathbb{E}(T_{k_n})}{k_n}=\lim_{n\to\infty}\frac{1}{k_n}\sum_{m=1}^{k_n}\mathbb{E}(Y_m)=\lim_{m\to\infty}\mathbb{E}(Y_m)=\mu,$$

and hence

$$\frac{T_{k_n}}{k_n} \to \mu \quad a.s. \tag{3}$$

3. The last step is to consider intermediate terms. For $k_n \le m \le k_{n+1}$, by Step 1, $X_k \ge 0$, so $Y_k \ge 0$ and hence $T_{k_n} \le T_m \le T_{k_{n+1}}$, then

$$\frac{T_{k_n}}{k_{n+1}} \le \frac{T_m}{m} \le \frac{T_{k_{n+1}}}{k_n},$$

i.e.

$$\frac{T_{k_n}}{k_n} \cdot \frac{k_n}{k_{n+1}} \le \frac{T_m}{m} \le \frac{T_{k_{n+1}}}{k_{n+1}} \cdot \frac{k_{n+1}}{k_n}. \tag{4}$$

Since

$$\frac{\alpha^{n+1}-1}{\alpha^n} \le \frac{k_{n+1}}{k_n} = \frac{[\alpha^{n+1}]}{[\alpha^n]} \le \frac{\alpha^{n+1}}{\alpha^n - 1},$$

we have $k_{n+1}/k_n \to \alpha$ as $n \to \infty$, then (3) and (4) implies

$$\frac{\mu}{\alpha} \le \liminf_{m\to\infty} \frac{T_m}{m} \le \limsup_{m\to\infty} \frac{T_m}{m} \le \alpha\mu, \quad a.s.$$

let $\alpha \to 1^+$, we have

$$\lim_{m\to\infty} \frac{T_m}{m} = \mu \quad a.s.$$

□